Dynamique(s) de descente pour l'optimisation multi-objectif

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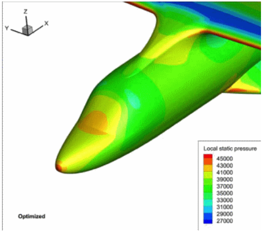
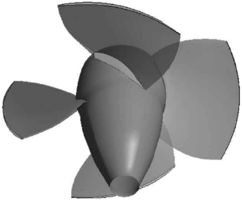
Journées SMAI-MODE
24 Mars, 2016

In engineering, decision sciences, it happens that various objective functions shall be minimized simultaneously: $f_1, ..., f_m : H \longrightarrow \mathbb{R}$





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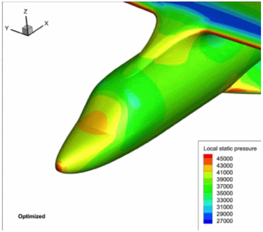

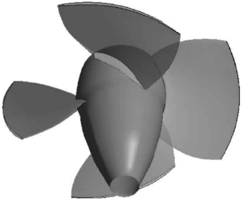

$\longrightarrow$ Needs appropriate tools: multi-objective optimization.

The multi-objective optimization problem

Let $F = (f_1, ..., f_m) : H \to \mathbb{R}^m$ locally Lipschitz, H Hilbert.

$$\text{Solve} \;\; \text{MIN}\; (f_1(x), ..., f_m(x)) \;:\; x \in C \subset H \text{ convex.}$$

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We consider the usual order(s) on $\mathbb{R}^m$:

$$a \preceq b \Leftrightarrow a_i \leq b_i \text{ for all } i = 1, ..., m,$$
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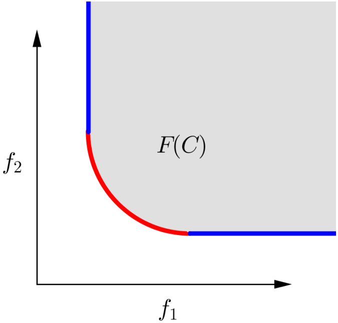
x is a Pareto point if
$\nexists y \in C$ such that $F(y) \precneqq F(x)$

x is a weak Pareto point if
$\nexists y \in C$ such that $F(y) \prec F(x)$



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- genetic algorithm $\longrightarrow$ no theoretical guarantees.

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- genetic algorithm $\longrightarrow$ no theoretical guarantees.
- scalarization method:

$$\bigcup_{\theta \in \Delta^m} \underset{x \in H}{\text{argmin}}\ f_\theta(x) \subset \{\text{weak Paretos}\} \subset \{\text{Paretos}\},$$

where Δ^m is the simplex unit and $f_\theta(x) := \sum_{i=1}^{m} \theta_i f_i(x)$.

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We are going to present a method which:

- generalizes the gradient descent dynamic $\dot{x}(t) + \nabla f(x(t)) = 0$,
- is *cooperative*, i.e. all objective functions decrease simultaneously,
- is independent of any choice of parameters.

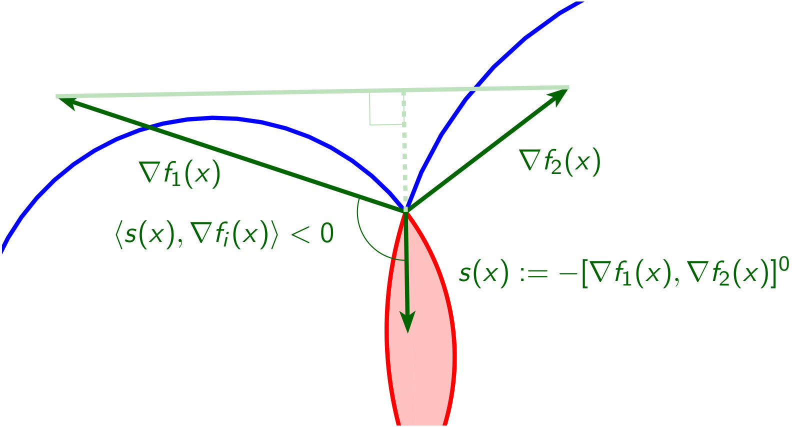
Single objective optimization:

$$x_{n+1} = x_n + \lambda_n d_n,$$

where d_n satisfies $df(x_n; d_n) < 0$ (e.g. $d_n = -\nabla f(x_n)$).

Multi-objective optimization:
Can we find d_n such that $df_i(x_n; d_n) < 0$ for all $i \in \{1, ..., m\}$?

Cornet (1981)


$\nabla f_1(x)$
$\nabla f_2(x)$
$\langle s(x), \nabla f_i(x) \rangle < 0$
$s(x) := -[\nabla f_1(x), \nabla f_2(x)]^0$

Multi-objective steepest descent

Let $F = (f_1, ..., f_m) : H \longrightarrow \mathbb{R}^m$ locally Lipschitz, $C = H$ Hilbert.

Definition

For all $x \in H$, $s(x) := -\left(\mathrm{co}\,\{\partial^c f_i(x)\}_{i=1,...,m}\right)^0$ is the (common) steepest descent direction at x.

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Remarks in the smooth case

- If $m = 1$ then $s(x) = -\nabla f_1(x)$.

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- At each x, $s(x)$ selects a convex combination:

$$s(x) = -\sum_{i=1}^{m} \theta_i(x) \nabla f_i(x) = -\nabla f_{\theta(x)}(x) \text{ where } f_{\theta(x)} = \sum_{i=1}^{m} \theta_i(x) f_i.$$

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- $s(x)$ is the steepest descent:

$$\frac{s(x)}{\|s(x)\|} = \underset{d \in \mathbb{B}_H}{\mathrm{argmin}} \left\{ \max_{i=1,...,m} \langle \nabla f_i(x), d \rangle \right\}.$$

Algorithm:

$$x_{n+1} = x_n + \lambda_n s(x_n).$$

Studied in the 2000's by Svaiter, Fliege, Iusem, ...

Continuous dynamic:

$$\text{(SD)} \quad \dot{x}(t) = s(x(t)),$$

$$\text{i.e.} \quad \text{(SD)} \quad \dot{x}(t) + \left(\mathrm{co}\,\{\partial^c f_i(x(t))\}_i\right)^0 = 0$$

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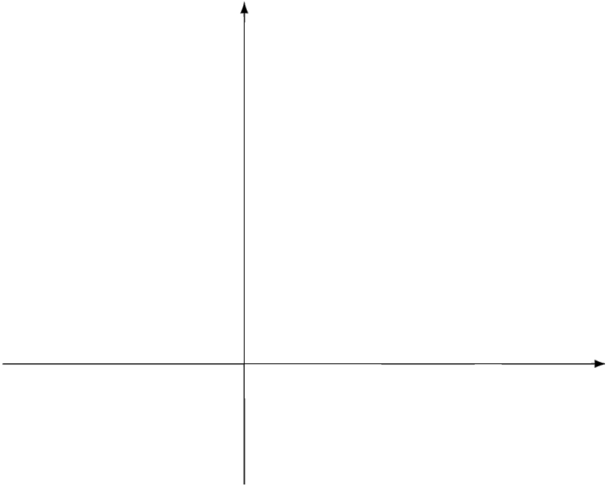

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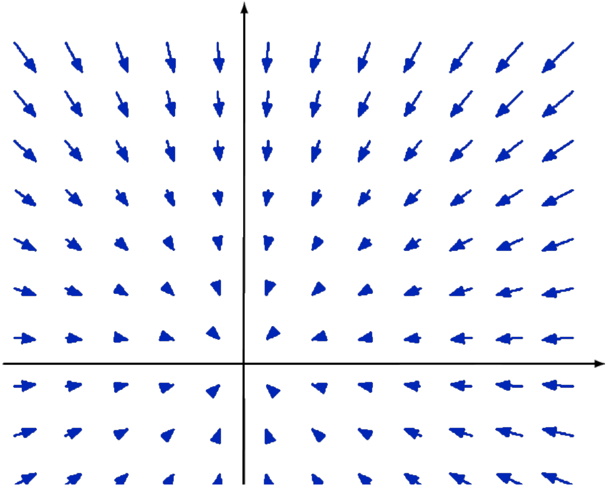

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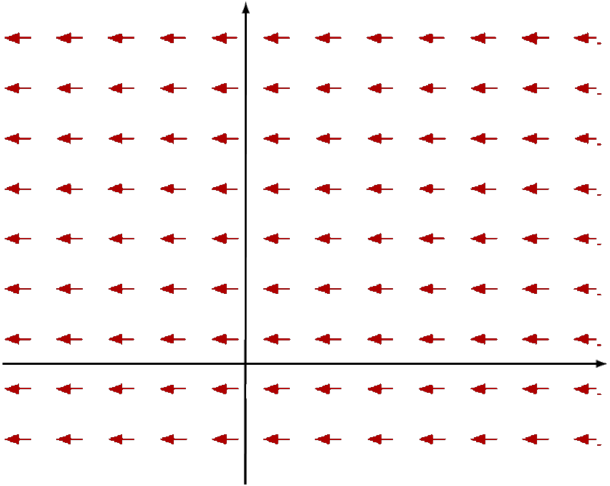

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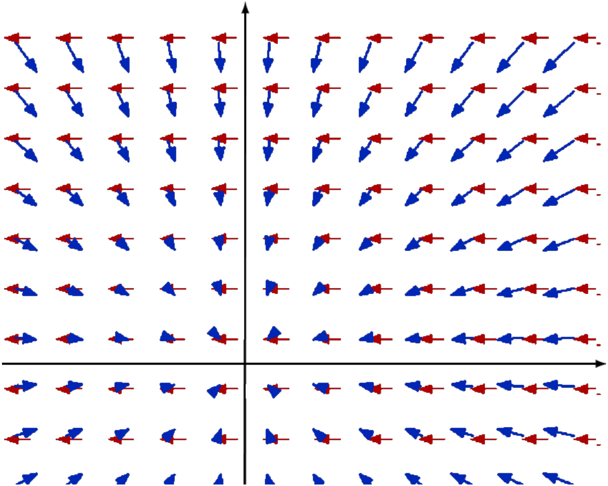

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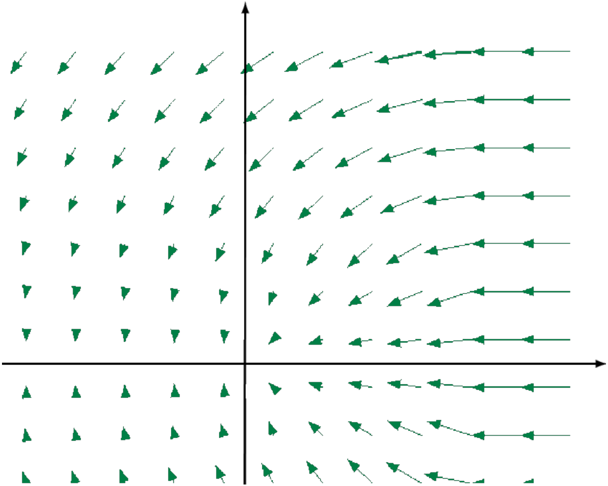

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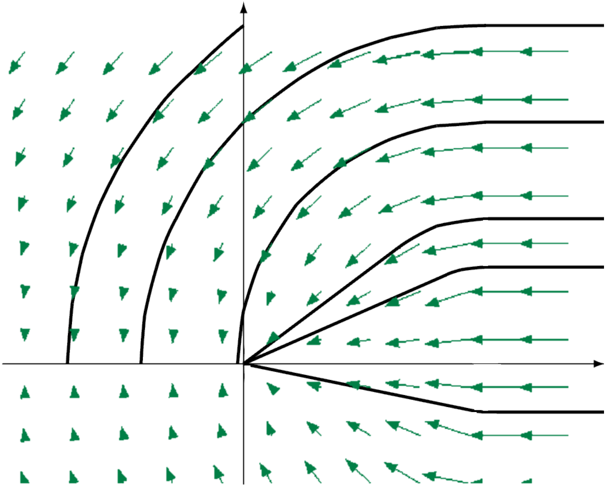

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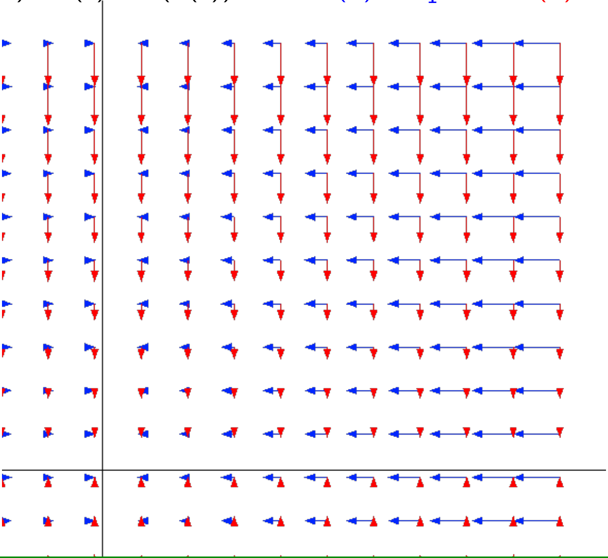

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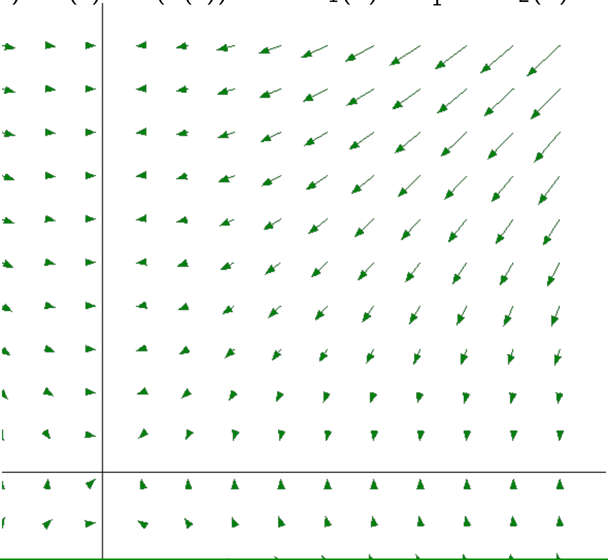

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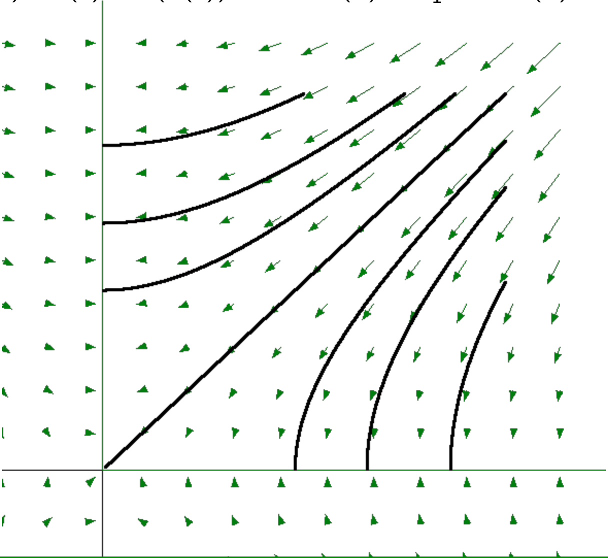

A cooperative dynamic

Let $x : \mathbb{R}_+ \longrightarrow H$ be a solution of (SD) $\dot{x}(t) = s(x(t))$.
For all $i = 1, ..., m$, the function $t \mapsto f_i(x(\cdot))$ is decreasing.

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Convergence in the convex case

Assume that the objective functions are convex. Then any bounded trajectory weakly converges to a weak Pareto point.

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Existence in the convex case

Suppose that H is finite dimensional. Then, for any initial data, there exists a global solution to (SD).

- In case of convex constraint $C \subset H$:

$$\text{(SD)} \quad \dot{x}(t) + \left(N_C(x(t)) + \operatorname{co}\left\{\partial^c f_i(x(t))\right\}_i\right)^0 = 0.$$

How to discretize it properly?

- In case of convex constraint $C \subset H$:

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- Uniqueness? Yes, if $\{\nabla f_i(x(\cdot))\}_{i=1,\dots,m}$ are affinely independants.

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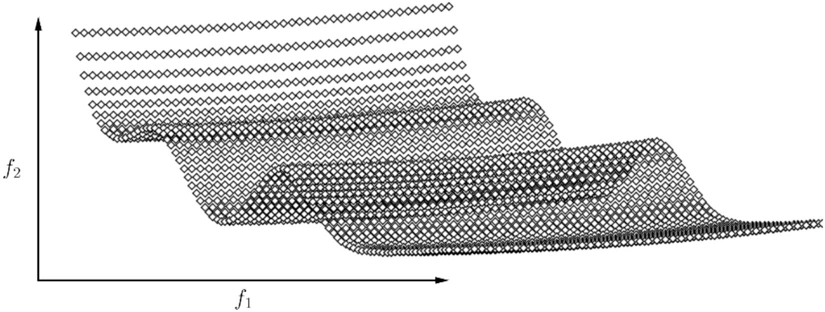
- Uniqueness? Yes, if $\{\nabla f_i(x(\cdot))\}_{i=1,\dots,m}$ are affinely independants.
- Convergence to Pareto points? Guaranteed by endowing $\mathbb{R}^m$ with a different order (but some of the Paretos might be lost in the operation).

$$f_1(x,y) = x + y$$
$$f_2(x,y) = x^2 + y^2 + \frac{1}{x} + 3e^{-100(x-0.3)^2} + 3e^{-100(x-0.6)^2}$$
$$(x,y) \in C = [0.1,1]^2$$



Plot of $F(C)$, $F = (f_1, f_2) : C \longrightarrow \mathbb{R}^2$.

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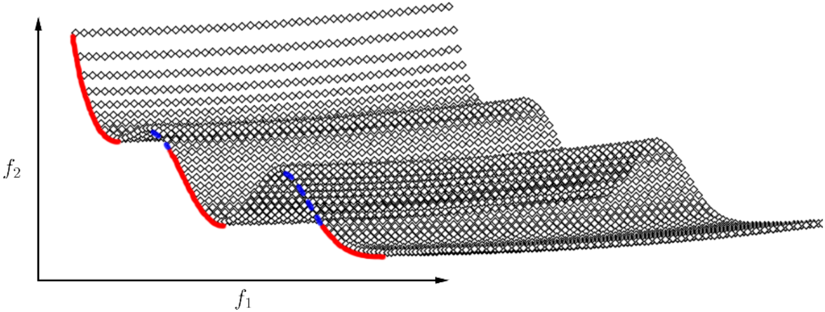

Plot of $F(C)$, $F = (f_1, f_2) : C \longrightarrow \mathbb{R}^2$ and its pareto front.

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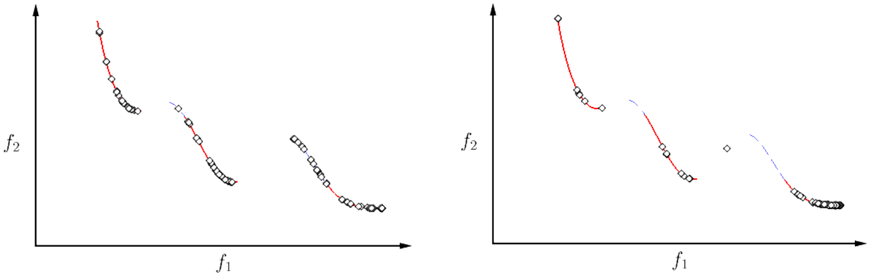

Gradient method (Right) vs Scalar method (Left). 100 samples.

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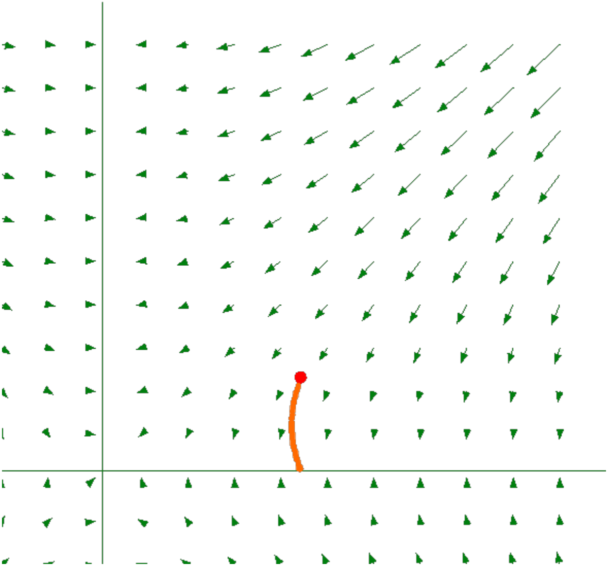
$\rightarrow$ Tikhonov regularization

$$\dot{x}(t) - s(x(t)) + \varepsilon(x(t) - x_d) = 0, \varepsilon > 0.$$



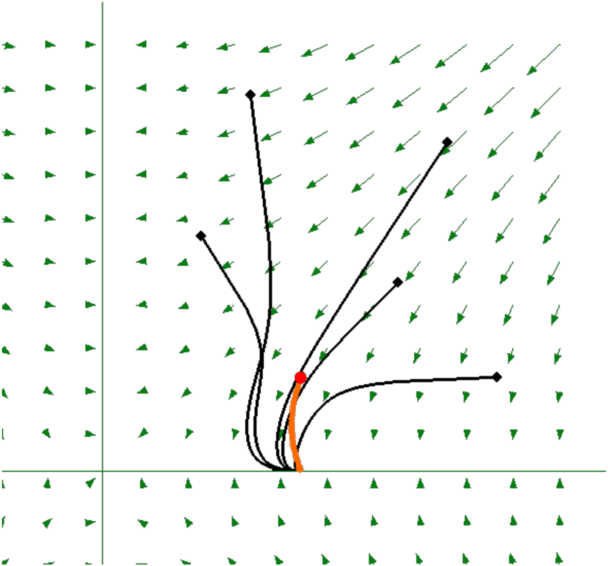
Can we select, among the weak Paretos (= the zeros of $x \mapsto s(x)$) the closest to a desired state?

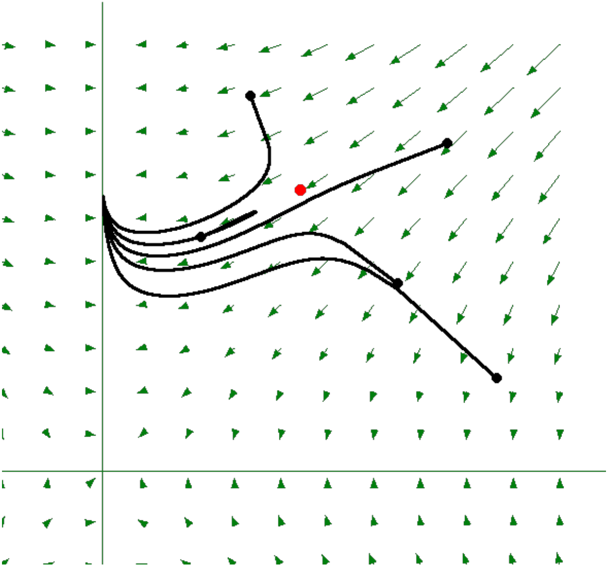
$\rightarrow$ Diagonal Tikhonov regularization

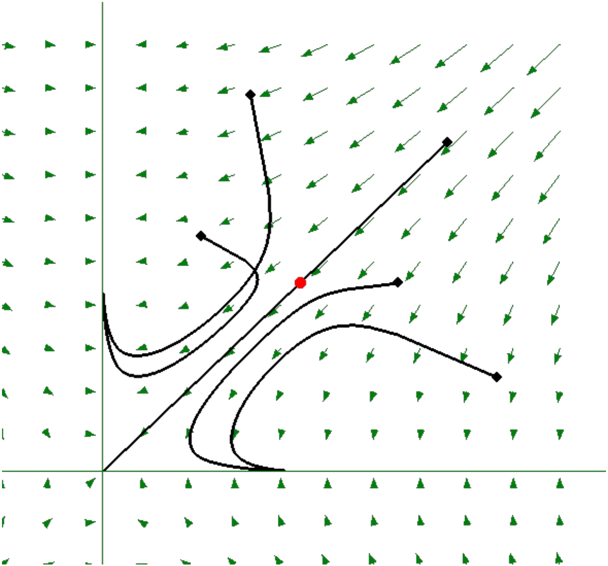
$$\dot{x}(t) - s(x(t)) + \varepsilon(t)(x(t) - x_d) = 0,$$

$$\varepsilon(t) \downarrow 0, \int_0^{\infty} \varepsilon(t)\ dt = +\infty.$$

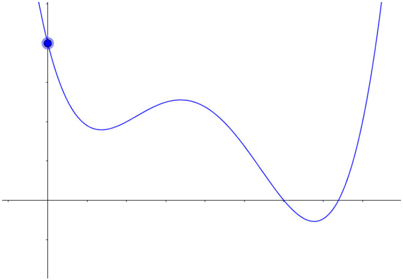
See the works of Attouch, Cabot, Czarnecki, Peypouquet (...) in the monotone case.

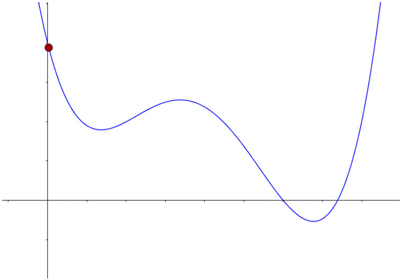






What about inertial dynamics?





$$\dot{x}(t) + \nabla f(x(t)) = 0$$

$$x_{n+1} = x_n - \lambda \nabla f(x_n)$$

$$\ddot{x}(t) + \gamma \dot{x}(t) + \nabla f(x(t)) = 0$$

$$x_{n+1} = y_n - \lambda \nabla f(y_n)$$
$$y_{n+1} = x_{n+1} + (1-\gamma)(x_{n+1} - x_n)$$

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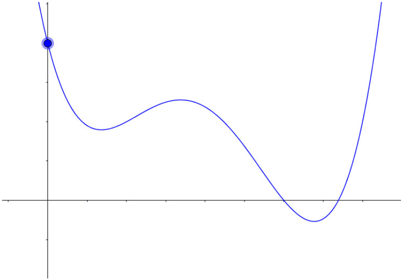

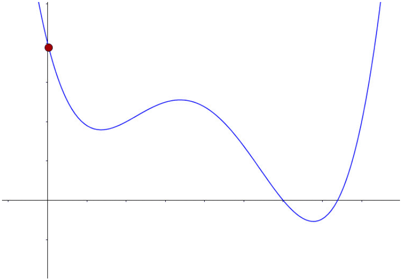

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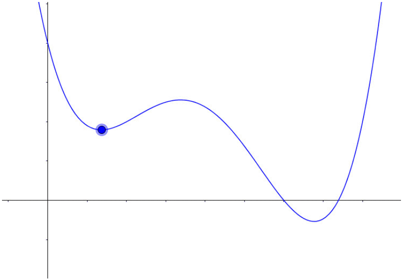

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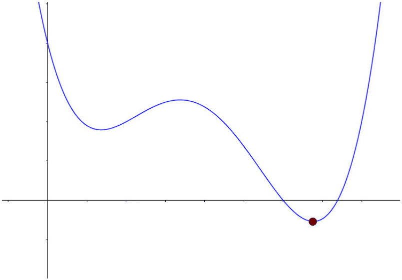


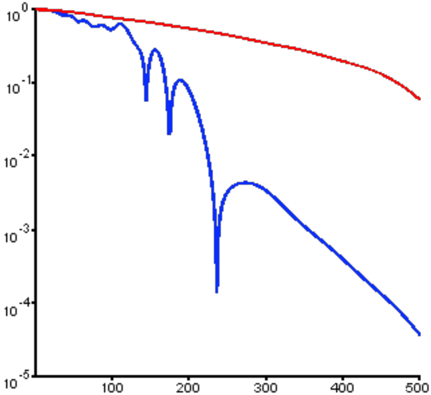
$$\ddot{x}(t) + \gamma \dot{x}(t) + \nabla f(x(t)) = 0$$

Inertia promotes

- Faster trajectories (varying γ),
- Exploratory properties.

Convergence rates : empirical observation

$$f_1(x)=\left(\sum_{i=1}^{10} x_i^2-10cos(2\pi x_i)+10\right)^{\frac{1}{4}},\ f_2(x)=\left(\sum_{i=1}^{10}(x_i-1.5)^2-10cos(2\pi(x_i-1.5))+10\right)^{\frac{1}{4}}$$



Convergence rate of $\|F(x^n) - F(x^\infty)\|_\infty$:
Steepest Descent vs Inertial Steepest Descent

Inertial (multi-objective) Steepest Descent

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Example: $f_1(x) = \|x\|^2$ and $f_2(x) = x_1$.

Let $f_1, ..., f_m$ be smooth, with L-Lipschitz gradient.

$$\text{(ISD)} \; m\ddot{x}(t) = -\gamma \dot{x}(t) + s(x(t)).$$

Assume that $\gamma \geq L$.

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Convergence in the convex case

Let $f_1, ..., f_m$ be convex. Then, any bounded trajectory weakly converges to a weak Pareto point.

Conclusion

The steepest descent provides a flexible tool once adapted to multi-objective optimization problems.

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Open questions:

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  (the set of weak Paretos is non-convex).
- Having convergence rates for first and second-order dynamics (the critical values are not unique).

Thank you for your attention !